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Al-Khwârizmî, Ibn Turk, and the Liber Mensurationum: on the Origins of Islamic Algebra

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AL-KHWÂRIZMÎ, IBN TURK, AND THE LIBER MENSURATIONUM: ON THE ORIGINS OF ISLAMIC ALGEBRA*

By JENS HØYRUP**

Abstract

A close investigation of Old Babylonian second-degree algebra shows that its method and conceptualization are not arithmetical and rhetorical, as is *grosso modo* the *al-jabr* presented by Al-Khwârizmî. Instead, it appears to be based on a "naive" geometry of areas very similar to that used by Ibn Turk and Al-Khwârizmî in their justifications of the algorithms used in *al-jabr* to solve the basic mixed second-degree equations.

This raises in a new light the question whether the early Islamic use of geometric justifications was a graft of Greek methods upon a "sub-scientific" mathematical tradition, as often maintained, or the relation of early Islamic algebra to its sources must be seen differently.

Now, the *Liber Mensurationum* of one Abû Bakr, known from a twelfth-century Latin translation, refers repeatedly to two different methods for the solution of second-degree algebraic problems: A basic method, may be identified as "augmentation and diminution" (*al-jam' wa'l-tafrîq?*), and another one labelled *al--jabr*, which coincides with al-Khwârizmî's use of numerical standard algorithms and rhetorical reduction. Since the *Liber Mensurationum* coincides in its phrasing and in its choice of grammatical forms with Old Babylonian texts, and because of peculiar details in the terminology and the mathematical contents of the text, it appears

* The following is a slightly revised version of my contribution to the International Symposium on Ibn Turk, Khwârizmî, Fârâbî. Beyrûnî, and Ibn Sînâ, Ankara, September 9-12, 1985. An abridged version of the article will be found in the Proceedings of the Symposium. My sincere thanks are due to Professor Aydın Sayılı, who invited me to the Symposium, and who insisted to have the full article published in the present journal.

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to represent a direct sub-scientific transmission of the Old Babylonian naive-geometric algebra, bypassing Greece as ~~wel~~ well as late Babylonian (Seleucid) algebra as known to us. This, together with internal evidence from Al-Khwârizmî's *Algebra* and Thâbit's Euclidean justification of the algorithms of *al-jabr*, indicates that Ibn Turk and Al-Khwârizmî combined *two* existing sub-mathematical traditions with a "Greek" understanding of the nature of mathematics, contributing thereby to the reconstruction of the subject as a scientific mathematical discipline.

An appendix discusses on basis of this new evidence the prehistory of the terms *al-jabr* and *al-muqâbala*. A second appendix presents another instance of very faithful transminssion of Old Babylonian methods and formulations to the sub-scientific mathematics of the Middle Ages, concerning the series 2^n. Appendix III contains a reprint of some key pages from Rosen's rare translation of Al-Khwârizmî's *Algebra*.

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General notes :

In all lexical questions and in problems concerning comparative Semitic philology, I build on the following dictionaries :

Hans Wehr, *A Dictionary of Modern Written Arabic*. Edited by J. Milton Cowan. Third Printing, Wiesbaden: Otto Harrassowitz, 1971.

Wolfram von Soden, *Akkadisches Handwörterbuch*. I-III. Wiesbaden : Otto Harrassowitz, 1965-1981.

Wilhelm Gesenius, *Hebräisches und Aramäisches Handwörterbuch über das Alte Testament* bearbeitet von, Frants Buhl. Sechszehnte Auflage. Leipzig : F.C.W. Vogel, 1915.

Since I read neither Arabic nor Hebrew and only the simple Akkadian of the mathematical texts, I may well have made philological mistakes, for which I apologize in advance. I will be grateful for corrections in this as in other domains.

For reasons of typographical commodity, I indicate long vowels by ^ instead of – in Arabic expressions. In order to avoid unnecessary errors Hebrew words are given without vocalization.

I. The Traditional State of the Problem

Since the discovery some fifty years ago that certain cuneiform texts solve equations of the second degree[1] the ideal has been close at hand that the early Islamic[2] algebra known from Al-Khwârizmî and his contemporary Ibn Turk continues and systematizes an age-old tradition. More recently, Anbouba (1978: 76ff) has also made it clear that the two scholars worked on a richer contemporary background than can be seen directly from their extant works.[3] In fact, the same richer tradition can be glimpsed, e.g., from some scattered remarks in Abû Kâmil's *Algebra* — cf., below, section VI.

Hitherto, a main argument for the assumption of continuity has been a reading of the Babylonian texts and desctiptions of purely numerical algorithms, analogous to the rules given by Al-Khwârizmî. To exemplify the similarity, we may first look at Al-Khwârizmî's rule for the case "Roots and Squares[4] are equal to Numbers",

[1] For brevity, I shall permit myself to use certain modernizing terms without discussion — "equation", "second degree", etc. I take up problem of anachronism in another context (1985a).

[2] I use the term "Islamic" in the sense of "beloging to the culture and society of [Medieval] Islâm". In this sense, Thâbit as well as the young Al-Samaw'al are "Islamic" mathematicians, although they were not Muslims. I have chosen the term instead of the alternative "Arabic mathematics" because I consider Islâm and not the Arabic language the unifying force of the culture in question — cf., my (1984, esp. pp. 29f).

[3] This conclusion holds good even if the ascription of a *Kitâb al-jabr wa'l-muqâbala* to Sahl ibn Bishr (p. 79, on the faith of the Flügel-edition of Al-Nadîm's *Fihrist* must probably be considered erroneous. Cf. Saidan 1978: 23, GAS V, 245, or Suter 1892: 62f, n. 166.

[4] "Square" is Rosen's translation for *mâl*, literally "fortune" or "wealth", cf., below.

illustrated by "one square, and ten roots of the same, amount to thirty-nine dirhems":

> You halve the number of the roots, which in the present instance yields five. This you multiply by itself: the product is twenty-five. Add this to thirty-nine; the sum is sixty-four, Now take the root of this, which is eight, and subtract from it half the number of the roots, which is five; the remainder is three. This is the root of the square which you sought for; the square itself is nine. (Rosen 1831: 8)

A similar Babylonian problem (BM 13901, No. 1) was translated as follows by Thureau-Dangin (I replace the sexagesimal numbers by
notation current ~~notaion~~) :

> J'ai additionné la surface et le côté de mon carré: 3/4.* Tu poseras 1, l'unité. Tu fractionneras en deux: 1/2. Tu croiseras 1/2 et 1/2: 1/4. Tu ajouteras 1/4 à 3/4: 1. C'est le carré de 1. Tu soustrairas 1/2, que tu as croisé, de 1 : 1/2, le côté du carré.
>
> (TMB, 1)

Apart from the point that Al-Khwârizmî identifies the numbers used inside the algorithm by reference to the statement of the problem, while our Old Babylonian scribe identifies the 1/2 at its second occurence by reference to the first occurence in the procedure, the styles of the two treatments appear indeed to be quite similar. The tradition seems to be one of correct but unjustified and unexplained numerical computation, and a main innovation of the two early Islamic algebrists appears to be their introduction of "naive-geometric" justifications for the traditional standard procedurs (cf., appendix III).

In terms which I shall use recurrently below, it looks from the traditional translations as represented by my extract from TMB as if the *basic conceptualization* —i.e. the ontological status given to the fundamental entities used to represent the various concrete quantities dealt with in real or faked practical problems (be it numbers found in the tables of reciprocals, areas of fields, or prices) — was arithmetical: The "area" and the "side" of the square are, in this

* This and similar example of its kind in the following pages should be read as a fraction (or, a number followed by a common fraction).

traditional interpretation, nothing but names indicating the arithmetical relations between the powers of an unknown number, as it is the case in Diophantos' *Arithmetica*. Similarly, the *procedure* seems to be arithmetical — as it is also the case in Diophantos and in normal Islamic and Western "rhetorical" algebra. (In contrast, Al-Khwârizmî's and Ibn Turk's above-mentioned justifications are geometrical according to their *procedure*, although the *conceptualization* is arithmetical even here, the square and its side being thought to *represent* the numbers *mâl* and *jadr*, "wealth" and "root", i.e., unknown and its square root).

II. A New Interpretation of Old Babylonian Algebra

The above scenario for the development from Babylonian to early Islamic algebra is challenged by the results of a close investigation of the procedures and the basic conceptualization of Old Babylonian algebra in which I have been engaged for some years.[5,6] Close attention to the structure and use of the terminology shows, together with various other considerations, that the traditional reading of the texts provides us with a mathematically homomorphous but not with a correct picture: The lenghts and areas of the texts have to be accepted at face value, in agreement with a geometric conceptualization. Similarly, the procedure turns out to be one of "naive", constructive geometry of areas, very similar to but more primitive than the jusitifications found in Al-Khwârizmî and Ibn Turk.[7]

In order to support these statements I shall translate and explain three Babylonian problems, using the more precise meanings of terms which have come out of my investigation.

[5] First briefly communicated (in Danish) in my (1982). Later preliminary presentation in my (1984a), revised as (1985). MS in progress (1985a).

[6] In the first instance, I speak only of the Old Babylonian algebra texts, dating from c. 1800 B.C. to c. 1600 B.C. In section III I shall return to the question of the next documented phase of Babylonian algebra, the Seleucid texts (3d to 2nd century B.C.).

[7] So, the texts distinguish four different "multiplicative" operations and two different "additions". In the arithmetical interpretation these distinctions are both aimless and meaningless; in a geometrical interpretation the operations are different.

Let us first have a second look at the text quoted above from Thureau-Dangin (BM 13901 — translated this time from the transliterated text in MKT III, 1):

> The surface and the square-line I have accumulated: 3/4. 1 the projection you put down. The half of 1 you break, 1/2 and 1/2 you make span [a rectangle, here a square], 1/4 to 3/4 you append : 1, makes 1 equilateral. 1/2 which you made span you tear out inside 1: 1/2 the square-line.

The terminology is awkward, and must be so in order to render if only imperfectly a structure of concepts and operations different from ours. The "square-line" (*mitḫartum*) designates a square ~~in-~~ id-tified by (and hence with) the length of its side (as we have identified the figure with its area since the Greeks). The term means "that which confronts [its equivalent]" and derives from *maḫārum*, a word which is close to Arabic *qabila* in its total range of connotations. You "append" (*waṣābum*) x to y when performing a concrete (not abstract-arithmetical) addition in which the entity x conserves its identity (as a capital conserves its identity even when the bank adds the interests of the year), while you "accumulate" (*kamārum*) them in a more abstract addition where both addends loose their identity (apparently, the "accumulation" designates a real addition of measuring numbers). The "projection" (*wāṣītum*) is the width 1 which from a line of length x makes a rectangle of area x·1 = x. To "put down" translates *šakānum*, an all-purpose-term close to English "to put" or "to place" or to Arabic *waḍa'a*. To "break" (*ḫepûm*) is used with general division by 2). Two lines are "made span" (*šutākulum*) when a rectangle is created ("built" is the Babylonian expression — *banûm*, cf., Arabic *banâ*). The "equilateral" is another (Sumerianizing) term for the quadratic figure (a verb meaning "to be equal"), and the phrase "x makes y equilateral" is used to tell that y is the side of a square of area x. To "tear out" (*nasāḫum*) is a process of concrete, identity-conserving subtraction, the inverse of "appending".

With these explanations in mind, should be able to follow the procudure on Figure 1. Firstly the "projection" is placed projecting from one of the sides of the square. Next it is "broken" (together with the whole appurtenant rectangle), and the outer part moved so that the two "span" a square (dashed line in the third step) of

area 1/2. 1/2 = 1/4, which is appended to the gnomon resulting from the displacement of the broken-off rectangle. This larger square then has an area 1/4+3/4 = 1, and hence a side 1. The broken-off and displaced 1/2, which is part of this side 1, is "torn out" from it, leaving back the required "square-line".

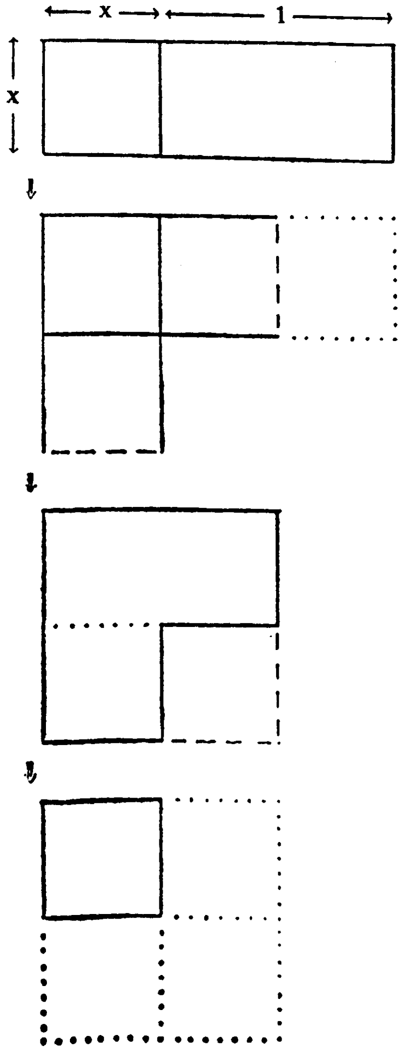


Figure 1. The geometrical interpretation of BM 13901 No. 1. Cf., Ibn Turk's figure in Sayılı 1962:163, and Al-Khwârizmî's in Rosen 1831:16 (below, appendix III).

If we compare this with Al-Khwârizmî's second variant of the justification of the case "a Square and ten Roots are equal to thirtynine Dirhems" (see appendix III), we find a very close agreement. Problem No. 23 of the same Old Babylonian tablet provides us with a parallel to his first variant, where 10·x are distributed equally along the four edges of the square x·x (MKT III, 4f; translated with sexagesimal numbers) :

> The surface of the four fronts and the surface I have accumulated: 0; 41, 40. 4, the four fronts, you inscribe. The reciprocal of 4, 0;15. 0;15 to 0;41, 40 you raise: 0; 10, 25 you inscribe. 1 the projection you append: 1; 10, 25 makes 1; 5 equilateral. 1 the projection which you appended you tear out: 0; 5 you double until twice: 0; 10 nindan confronts itself.

The translation calls for a few extra commentaries. To "raise" (*našûm*) is a term used when a concrete magnitude is to be calculated by multiplication (basically, it appears to refer to an argument by proportionality). To "double" (*eṣēpum*) involves repetition two or eventually more times (cf., Arabic *ḍiʿf*, which derives from the same root). The nindan is the basic unit of length (of value ca. 6 m.). Apart from single words, finally, it shall be emphasized that the grammatical construction used in the beginning makes it indubitably clear that *the* four fronts and not just 4 times the side are meant.

Let us now follow the text on Figure 2. The "surface of the four fronts" and the "projection" further down makes it clear that we have to begin with a cross-form configuration, as shown at the top. The multiplication by 1/4 (=0; 15) is shown next: One fourth of the cross is considered alone. The square on the "projection" (identified as a geometric picture with its side, the "projection" itself) is "appended", transforming the gnomon into a square, the area of which is found to be 1; 10, 25. Hence

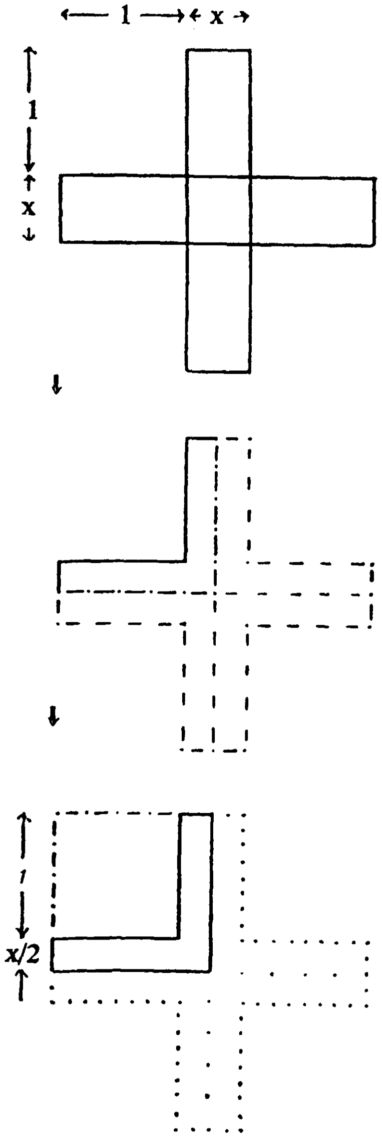


Figure 2. The geometrical interpretation of BM 13901, No. 23., cf. Al-Khwârizmî's Figure in Rosen 1831:15.

the side (that which 1; 10, 25 "makes equilateral") must be 1; 5. This side was composed by "appending" the "projection" to half the front; so, the "appended" 1 is torn out, and the remaining 0; 5 is doubled (repeated concretely, not "raised to 2"),which gives us that front which "confronted itself" in the original square.

It should be added that the problem is unique among published Babylonian texts. It stands towards the end of the tablet, among the complicated variations, far from the standard types of its beginning. I have the feeling that the problem may already be archaic in the Old Babylonian context : A sub-scientific practitioners' environment might easily suggest this type of recreational problem, and it might then inspire the formulation of more general second-degree equations in a systematizing school environment;[8] the reverse movement is, if not impossible, less probable–especially in view of the fact that the same problem type turns up in Medieval Islâm precisely in mensuration (*misâḥa*) texts.

A third problem (AO 8862, No. 1; in MKT I, 108f) is more complicated. For easy reference, I divide it into sections.

A Length, width. Length and width I have made span: A surface I have built. I turn around. So much as that by which the length exceeds the width I have appended to the inside of the surface: 183. I turn back. Length and width accumulated: 27. Length, width, and surface how much?

B	27	183	accumulation	
	15	length	180	surface
	12	width		

C You, by your making, append 27, the accumulation of length and width, to the inside of 183: 210. Append 2 to 27: 29.

D Half of it, that of 29, you break: 14 1/2. <14 1/2 and 14 1/2 you make span>. 14 1/2 times 14 1/2, 210 1/4. From the inside of 210 1/4 you tear out 210: 1/4 the remainder. 1/4 makes 1/2 equilateral. Append 1/2 to the first 14 1/2: 15 the length. You tear out 1/2 from the second 14 1/2: 14 the width.

[8] I discuss the role of the school for the development and character of Mesopotamian mathematics in my (1985b: 7-17).

E 2 which you have appended to 27 you tear out from 14, the width: 12, the true width.

F 15 the length, 12 the width make span: 15 times 12, 180 the surface. By how much does 15, the length, exceed 12, the width: It exceeds by 3; append it to the inside of 180, the surface, 183 the surface.

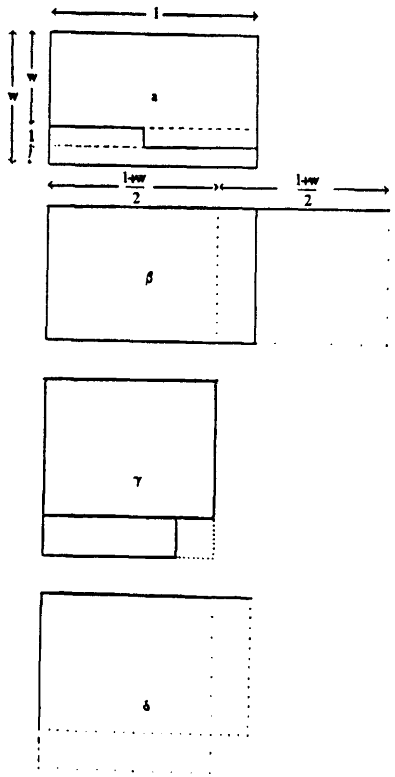


Figure 3. The geometrical interpretation of AO 8862 No. 1. Distorted proportions. Cf., Ibn Turk's identical figure in Sayılı 1962: 164, and Al-Khwârizmî's in Rosen 1831:18.

The "length, width" in the beginning tell that the problem deals with a rectangle. The "turning around" and "turning back" in A mark sections of the statement. B tell in advance the dimensions of the figure (and so, the procedure part tells the student *how to obtain* these results known in advance). The "times" of D (and F) translates a-rà, the multiplicatory term of the multiplication tables (meaning literally "steps of"). The insertion < > in D is made on the faith of parallel passages (among which one in F).

We may now follow the text on Figure 3. In the first section of the procedure (C), the known sum of length (l) and width (w) is "appended" "to the inside of" 183, yielding (when the one-dimensional lengths are provided with an implicit "projection") a rectangle of length $l = 15$, width $W = w + 2 = 14$, and area 210 (α).

Through this geometric "change of variable" the problem is reduced to one of the standard problems of Babylonian algebra, which is solved in section D: The sum of length and width is bisec-

ted (β), and its halves are "made span" a square of area $14\ 1/2 \cdot 14\ 1/2 = 210\ 1/4$ (γ). The full-drawn gnomon inside square, which is equal to the rectangle and hence to 210, is "torn out", leaving the small square (lower right corner) of area 1/4 and hence of side 1/2. Finally, this 1/2 is "appended" to the horizontal side of the large square, yielding the length l of the rectangle, and "torn out" from its vertical side, yielding its width W (δ) — the width, that is, of the augmented rectangle.

In section E, the original ("true") width w is found by subtraction. Section F, finally, controls the correctness of the results.

By comparison with Al-Khwârizmî's *Algebra* one finds that the procedure of section D is exactly the one given there to justify the algorithm for the case "a Square and twenty-one Dirhems are equal to ten Roots" (Rosen 1831: 16-18). The same procedure is given by Ibn Turk (Sayılı 1962: 163f), while Abû Kâmil uses a slightly different figure apparently inspired by *Elements* II. 5 (Levey 1966:44-46) — the proposition, indeed, to which Thâbit refers in *his* demonstration of the same matter (Luckey 1941: 106f).

III. *Seleucid Testimony*

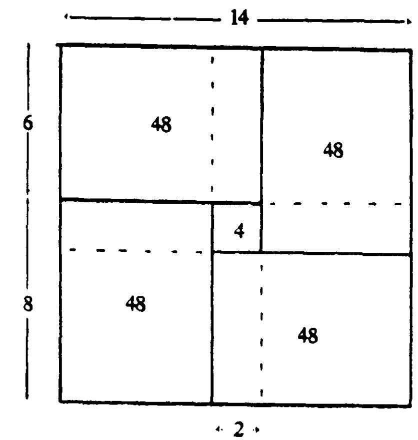


Figure 4. A geometric figure which will serve as "naive" justification for the calculational steps in BM 34568 No. 9 (and which, when diagonals are drawn in the rectangles, will demonstrate the Pythagorean theorem and a variety of derived identities through simple counting.

As stated above, the next phase of documented Babylonian algebra belongs in the Seleucid era. Since many changes can be seen in the texts to have taken place by then since the Old Babylonian period, and since these changes bear upon the question of continuity until early Islamic algebra, I shall indicate the style of this phase by translating a simple problem (BM 34568 No 9; translated after the text in MKT III, 15, as corrected in Von Soden 1964: 48a):

Length and width accumulated: 14, and 48 the surface. I do not know the name. 14 times 14, 196. 48 times 4, 192. Go up from 192 to 196: 4 remains. How much times how much shall I go in order to get 4: 2 times 2, 4. Go up from 2 to 14, 12 remains. 12 times 1/2, 6. 6 the width. Add 2 to 6, 8 the length.

The most conspicuous change is probably the completely arithmetical conceptualization of a problem which is formally presented as geometric. Numbers in mutual arithmetical relation are used to represent the geometric entities involved; subtraction and multiplication are thought of as counting procedures ("go up from X to Y"; "go X steps of Y"), and a square root is understood as the solution to the arithmetical equation $x \cdot x = A$.

Another change is found in the structure of the procedure. It is possible, and ideed plausible, that the procedure is still geometric — but in any case it is different from the Old Bablylonian procedure. The latter would find the semi-difference between the length and the width and would add it to and subtract it from their semi-sum. Here, the *total* difference is found, and added to their sum, result being then halved to yield the length, etc. The possible geometric argument is also made, so it looks, on a ready-made figure (see Figure 4) — the text contains no trace of constructive procedures. (It should be observed, however, that a constructive description of the same figure appears to be used to solve the Old Babylonian problem YBC 6504, No. 2, see my (1985:42ff); if really used, the figure need not have been a Seleucid invention).

IV. The Liber Mensurationum

(Bubnov An 11th (?) century (A.D.) manuscript (Bibliotheca Amploni-
1899:391ff) anae, No. 362) contains a problem, which according to Cantor (1875: 104) may go back in its Latin version to the fourth century A.D., and which appears to have been translated from an Alexandrian source. It deals with a right triangle, of which the hypotenuse and the area are known. It leads to a second-degree equation, which is solved by the Seleucid method, — and indeed, the problem itself is closely related to the sort of problems dealt with in the Seleucid tablet just quoted. So, the Alexandrian knowledge of second-degree equations (as also testified in Heron's *Geometrica*, Heiberg, 1912:380) appears to be more closely related to Seleucid than to Old Babylonian mathematics (and conversely, Seleucid practical geometry seems closer to "Heronian" geometry than to Old Babylonian surveying practices, cf. VAT 7848 No 3, in MCT, 141). This could lead to the idea that a continuous development goes from Old Babylonian texts over Seleucid and Alexandrian applied mathematics to the earlier Middle Ages.

The more astonishing is the contents of a *Liber Mensurationum*, "Book on *'ilm al-misâḥa*", "translated and abbreviated" by Gherardo of Cremona in the 12th century A.D. from Arabic into Latin, and written originally by an otherwise unidentified Abû Bakr (cf., GAS, V, 389f; Busard 1968 contains a critical edition).

The contents of the treatise is of evidently mixed origin. Its second half, dealing with trapezia, triangles, circle and circular sections, and finally with solids, has a strong Alexandrinian flavour. The first half (problems 1-64), dealing with square, rectangle, and rhomb, stands out for various reasons. It seems more archaic, and it is this part which I shall discuss here.

From various scattered references to "what precedes" it appears that the treatise was once a companion-piece to a presentation of *al-jabr*, *aliabra* in the Latin text instead of *algebra* (problems No 5, 9, 25, 26, etc.). The numbering of the basic mixed equations suggests that the companion has been in the Al-Khwârizmîan tradition.[9]

The treatise is important both because of the way it is organized "rhetorically" and for its mathematical substance. To illustrate this I shall translate some of its problems ("Hinduizing" verbal numerals

[9] Al-Khwârizmî's cases 4, 5 and 6 are numbered "first", "second" and "third" in Thâbit's *Rectification of the Cases of Algebra*, while Ibn Turk offers only description and no numbering at all. In view of other stylistic features of the translation (references to Divine good-will left in place) it seems implausible though not excluded that Gherardo has inserted a numeration which he knew from elsewhere (unless this is the point where he made the "abbreviation" claimed in the title); but admixture of Al-Khwârizmîan features during the Arabic transmission of the treatise is difficult to exclude, especially in view of a variety of clear corruptions of the text (No. 38 refers to No. 32 as immediately preceding and has furthermore taken up elements from some other problem; No. 57, which is repeated as No. 61, refers to No 58 as preceding; No. 16 is repeated as No. 18. Cf. also Busard (1968: 71) quoting and discussing Chasles.

Against the genuine character of the Al-Khwârizmîan influence speaks the use of the term *al-muqâbala* in a sense which is completely different from that of Al-Khwârizmî (see, below, appendix I).

since the Latin mix-up of verbal, Roman and Hindu numbers can hardly be original):

No 3 If he [i.e. a "somebody" presented in No. 1] has said to you: I have aggregated the side and the area [of a square], and what resulted was 110. How much is then each of its sides?

The method of this will be that you take the half of its side as half and multiply it with itself. 1/4 results, which you add to 110, which will be 110 1/4. You then take the root of this, which is 10 1/2, from which you substract the half, and 10 remain which are the side. See!

There is also another method to it according to *al-jabr*, which is that you take the side a thing and multiply it with itself, and what results will be the wealth, which will be the area. Then add this to the side as I said, and what results will be the wealth and a thing which equals 110. Do then as it preceded for you in *al-jabr*, which is that you halve the [coefficient of the] thing and multiply it in itself, and what results you add to 110, and you take the root of what comes out and subtract from it half the [coefficient of the] root. What then remains will be the side.

No 26 And if he has said to you: The area [of a rectangle] is 48, and the longer side adds the quantity of 2 over the shorter side; what then is each of the sides?

The method to find it will be that you halve the 2, and what results will be 1, which you multiply by itself, and 1 results. This same you then join to 48, and 49 results, of which you take the root which is 7, from which you subtract 1, and there remains 6 which is the shorter side. To this same then join 2, because his speech was: one side exceeds the other by the quantity of 2, and that which results will be 8. This then is the longer side.

But its method according to *al-jabr* is that you make the shorter side a thing. Then the longer will be a thing and 2, multiply hence a thing with a thing and with 2, whence wealth and 2 things will equal 48, which is the area. Do then according to what preceded for you in the fourth question [of *al-jabr*], and you will find it if it pleases God.

No 38 But if he has said to you: I have aggregated the longer and the shorter side and the area [of a rectangle], and what resulted was 62, while the longer side adds 2 over the shorter side; what is then each side?

The method to find this will be that you subtract 2 from 62, leaving back 60, and hence join 2 to half of the number of sides [sic!], from which 4 results. [...]

No. 45 But if he has said to you: I subtracted the longer side from the area [of a rectangle] and 40 remained, and the longer side adds 2 over the shorter side; what is then each side?

The method to find it will be, that you add 2 to 40, and it will be 42, which shall be kept in memory; then you subtract 1 from 2, and 1 remains. Take the half of it, which is 1/2, and multiply it with itself; and what results will be 1/4, which you shall join to the 42, and what results will be 42 1/4; take then its root, which is 6 1/2, and when the 1/2 is subtracted. 6 will remain which is the shorter side, over which the longer adds 2.

The method to find the same by *al-jabr* is simple.

Let us first look at the "rhetorical" aspect of the problems. The statements are formulated in the first person, preterite tense, by a "somebody". The same person and tense are used in the statement-part of Old Babylonian procedure texts,[10] and quite a few begin with the phrase *šumma kīam išāl-ka umma šū-ma*, "if somebody asks you thus:"[11] The beginning of the procedure-part, "the method to find it" etc., parallels the Old Babylonian *atta ina epēši-ka* "you, by your method", and similar expressions; the ensuing shift to the second person, present tense, alternating with the imperative, is also a repetition of a fixed Old Babylonian pattern, — and so are the references back to the speaker of the statement in the third person.

[10] But still, the excess of one side over the other is told in the present tense by Abû Bakr as already in Old Babylonia!

[11] E.g. all the 11 problems published in Baqir 1951. Other texts carry the shorter *šumma*, "if", but subsequent references to the statement in the procedure-part of the problem show this word to be an ellipsis for the complete construction. Still others carry even no "if", but *all* have the statement in the first person preterite, as a teacher or a "somebody" telling what he has already done.

More specifically, the construction of such references, "because his speach was" followed by a more or less literal quotation, corresponds to the Old Babylonian *aššum iqbû*, "because he said", equally followed by a quotation. Finally, the "which shall be kept in memory" of No. 45 (and other problems) corresponds to a recurrent Old Babylonian *rēš-ka likil*, "may your head retain".

None of these features are found in Seleucid texts. Taken singly, each of them might be explained as a random coincidence. It is, however, extremely implausible that so many structural features should be repeated together randomly. Even though no texts of a similar structure are known from the span between the end of the Old Babylonian period, ca. 1600 B. C., and the present work, we are forced to accept the existence of a continuous tradition during this immense span of time (and even of a *written* tradition, since purely oral transmission would hardly conserve the distinctions of tense and person in full sharpness). Furthermore, it appears that the Seleucid texts do not belong in the mainstream of this tradition. [12]

One element of the rhetorical framework has no Old Babylonian counterpart, *viz.* the "See!" which closes No. 3 and many other procedure-descriptions of the treatise. The Latin word is *intellige*, "understand"/"see", but as the text stands it presents no appeal to the understanding — Gherardo offers only prescriptions, no explanation or justification. Two reasons suggest, however, that the original term was one involving visually supported understanding.

Firstly, another text translated by Gherardo describing an Indian way to construct equilateral polygons tells us that "they have in their hands no demonstration of this but the device: *Intellige ergo.*" [13] This can, however, only refer to the Indian way to close the description of a method by the word "See!" and a drawing. [14] So, in one

[12] Since Seleucid mathematics teaching (and hence Seleucid mathematical texts) can probably be regarded as a spin-off from the highly specialized mathematical astronomy of the same period (not least because the tablets are presumably from Uruk, a main astronomical centre), it is indeed no wonder if it belongs on a branching and not on the mainstream of the algebraic tradition.

[13] The whole fragment is in Clagett 1984 : 600 f. "Device" translates *inventio*, which Clagett assumes on the basis of a marginal note to render *maujūd* or some other deivation from *wajada* (*ibid.* p. 474f, n. 12).

[14] So in several of the texts and commentaries translated by Colebrooke (1817).

place at least, Gherardo used *intelligere* as a (mis–) translation for an Arabic "See".

Secondly, the word is always to be found after the decsription of the basic procedure, the one which appears to descend directly from the naive-geometric Old Babylonian (cf., below); with procedures "according to *al-jabr*" it is strictly absent. [15] Furthermore, an *intellige* in No 2 corresponds to one of the few figures of the half of the treatise dealt with here. Finally, "understand it, whoever is able to!"

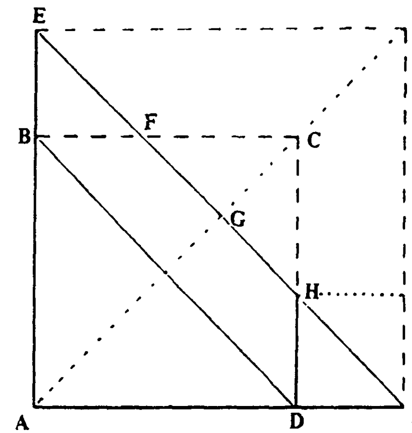


Figure 5. A possible geometric proof of the property of the square used in *Liber Mensurationum* No s. 16–17,

$$s = (d-s) + \sqrt{2.\ (d-s)^2}$$

The side of the larger square is made equal to the diagonal (d) of the smaller, and so its diagonal will be twice the smaller side (s). Now, EI = 2s = EH + HI = BD + HI = $d + \sqrt{2.\ DI^2} = d + \sqrt{2\ (d-s)^2}$, and hence the required identity.

The figure is related to others which yield the proportion contained in the Greek series of side–and diagonal – numbers (see, Hultsch in Kroll 1899-II, 393–400, and Bergh 1886). Indeed, the same proportion is contained in the above, *viz.*

DI : HI :: AD:BD, d–s: 2s–d:: s:d,

which is equivalent to the usual

S:D:: S+D: 2S+D

If S = d–s, D = 2s–d.

[15] I disregard an *intellige ergo et invenies* in the very end of No 50 (and hence after the al – *jabr* – procedure) for two reasons. Firstly a complete phrase "So understand, and you will find" must be considered different from the isolated word. Secondly the whole problem in question has the character of a joke and the closing sentence therefore that of a piece of irony : The preceding problem makes clear and explicit use of the fact that a rectangle where the difference between the diagonal (d) and the longer side (l) equals the difference between the longer and the shorter side (w) is proportional to the rectangle (d,l,w) = (5, 4, 3). Then comes No 50, dealing with such a rectangle with d = 10, and evertyhing should be simple. Instead, w is found by a truly dazzling procedure expressible as

$$w = \sqrt{[d^2 - (d/2)^2].\ (l - 1/5) + [1/2\ (1 - 1/5).\ (d/2)]^2} - 1/2\ .\ 4/5\ d/2$$

(where the square-root alone repsesents l), while 1 is then found by addition of 1/2 (d – w). Truly, "understand it, whoever is able to!"

ally, other figures belonging to the original treatise appear to have been lost in the process of transmission.[16]

On the limit between rhetorics and mathematical substance we find the mathematical vocabulary. Here it is interesting that the square is spoken of as *quadratum equilaterum et orthogonium*, "equilateral and rightangled quadrate", while the rectangle is considered a *quadratum altera parte longius*, a "quadrate longer at one side". Evidently, the Arabic original was written in a context where the word normally translated in the twelfth century as *square* (*viz.*, *murabba'*) was still understood in its general, pre-theoretical sense of quadrangle (cf. also below, section VII). This usage is in itself a suggestion of rather archaic, sub-scientific roots for the main framework of the (first half of the) treatise, in spite of the al-Khwârizmîan numbering of cases.

If we now turn to real mathematical substance, three questions turn up: The choice and formulation of problems; the distinction between the normal, apparently unnamed method, and the methods of *al-jabr*; and the character of the normal method (or methods). The *al-jabr*-methods are those familiar from Al-Khwârizmî and other sources and give rise to no fundamental questions.

Concerning the choice of problems, it was already observed by Busard that a number of these (including occasionally the numbers involved) coincide with Old Babylonian or Seleucid problems (like most other authors, Busard does not distinguish the two). Since the

[16] After No. 52, when the section on rectangles is said to end, comes an "and this is <its> form", referring obviously to a drawing (Fig. 2, p. 90 in Busard's edition). The closing sentence of the section on rhombs is the same (although the passage is evidently corrupt), and points to Figure 3 (p. 99). But in No 17, where a figure is badly needed to show why *the side* s and the diagonal d of a square fulfill the condition $s = (d-s) + \sqrt{2}\,.\,(d-s)^2$, the same sentence but no drawing is found.

The lost figure in that place seems to have corresponded to nothing known from Babylonian naive geometry; instead, it may have been related to one of the figures which can have given rise to the Greek recursive series approximating the ratio between side and diagonal of the square, cf. Figure 5. It may even lead to the same series itself. A Greek origin of the figure of the problem in question might be the reason that it is not referred to by *intellige*; indeed, all references to figures in the second, "Alexandrian" part of the treatise use the expression "and this is its form", with exception of Heronian treatments of circle and circular segment, which have the mixed "*intellige*, and this is its form".

number of simple second-degree algebraic problems and the number of e.g. simple pythagorean triples (important for the construction of problems on rectangles and rhombs) is restricted. I do not find this argument for direct connections very convincing — it would be utterly difficult to construct a statistical test of the hypothesis that the number of coincidences is greater than random. A few coincidences seem, however, to be difficult to explain away; so, it is far from evident that everybody knowing the Pythagorean theorem will stumble upon the identity $(l+w+d)^2-2A=2d\,.\,(l+w+d)$ which is in fact used both in the Seleuced tablet BM 34568 (No. 14, 17 and 18 — see MKT III, 16f and 21) and in *Liber Mensurationum* (No. 47) in analogous problems on a rectangle.[17]

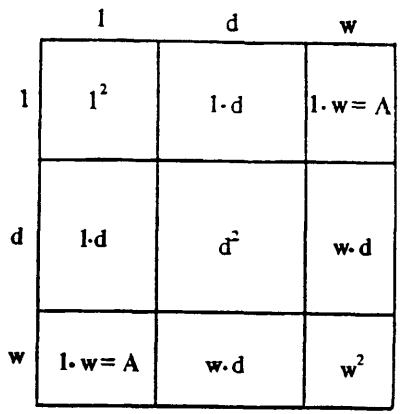


Figure 6. Diagram from which it can be seen that

$(l+w+d)^2 - 2A = 2d.\ (l+w+d)$

in a rectangle of length l, width w, diagonal d and area A (when $d^2 = l^2 + w^2$ is taken for granted). This identity is the basis for *Liber mensurationum*, No. 47. The diagram can also be used to show that

$(l+d)^2 + (w-d)^2 = (l+w+d)^2 + (l-w)^2$

which is the basis for No. 36.

What *can* be stated from problems alone is that the first half of the treatise is not just a *misâḥa*-handbook with the peculiarity that it makes use of algebraic methods: The majority of its problems would never occur in practical mensuration — instead, they can be obtained from such problems through interchange of known and unknown quantities; they are, in this sense, *algebraic* problems dressed in mensuration garments.

It can also be stated with great certainty that not all of the problems can derive from Babylonian sources. No. 51, dealing with

[17] What *is* evident is that play with certain figures inspired by the one shown in Figure 4 (which was presumably used in the Old Babylonian problem YBC 6504 No. 2, and perhaps in the Seleucid BM 34568 No. 9) would easily lead to the knowledge in question — see, Figure 6.

a rectangle in which $d:l::l:b$ and solving it apparently by reference to a division into extreme and mean ratio, could hardly have been formulated inside the conceptual framework of Babylonian mathematics. It seems related to early Greek (supposedly Pythagorean) geometry. The same may be the case of Nos. 16-17 (cf., note 13).

On the whole, however, the problems of the first part of the treatise are of a character reminding much of Old Babylonian and Seleucid mathematics, and which has little in common with Heronian and other Ancient material (and similar frail connection to Indian problem collections). This is true even for the problems dealing formally with a rhomb (Nos. 53-64), even though this figure is not known to have roused the algebraic interest of the Babylonians of any period.

A particular feature of the text is the interest in the sum or difference between the area and the four sides of a square or a rectangle. It is represented by no less than six problems (Nos. 4, 6, 9, 12, 43, and 46). In earlier mathematical traditions I know it only from the Old Babylonian BM 13901 No 23 (cf., above), and from the possible reflection in Al-Khwârizmî's *Algebra* (*if* this is earlier).

When it comes to solutions, the most striking feature is that the first description of the "method to find it" is followed by a second "method according to *aliabra*" in many problems. Since both procedures can apparently be regarded with equal right (or lack of right) as algebraic in more modern senses of that word, *aliabra* (and hence *al-jabr*) must have a more restricted sense,[18] to which Abû Bakr's counterposition can serve as a key.

In several cases (including No. 3 translated above) the numerical steps of basic and *al-jabr*-method are the same. The difference between the two must therefore be one of conceptualization or method, not one of algorithm (even though the algorithms are different in *most* cases). The explanation in No. 3 (and elsewhere) that the "wealth" is identical with the area shows us clearly that "wealth" and "root" are not to be understood *by themselves* as geometric quantities. *Al-jabr* is, according to the testimony of the text, concerned with the quantities "wealth", "root" and known number *connected arithmetically* (as it is also explained by Al-Khwârizmî — see Rosen 1831:6); its problems are formulated and reduced to fundamental cases by arithmetico-rhetorical methods (whence the "thing" turning up in the beginning and replaced later in the procedure -descriptions by "root") and the fundamental cases are solved by automatic algorithms, involving no justification, proof or just conceptualization of the intermediate steps. This in is fact, if we disregard his naive-geometric justifications, precisely the *al-jabr* known from Al-Khwârizmî.

[18] May be it was recognition of this restriction which led Gherardo to use a phonetic rendition of the Arabic term instead of the customary Latin *algebra* (with the exception of a slip in No 8 which suggests that the spelling is not that of the scribe who made the manuscript (cf., also Boncompagni 1851 : 439f).

The basic method must then be *something different.* As the descriptions stand, it looks as if itappeals even less to any sort of understanding; still whatever the meaning of *intellige*, be it "look" or "understand", this term involves *some* such appeal. Above, evidence speaking in favour of a visually supported understanding was discussed.

Further elucidation of the question may be achieved through investigation of Nos. 38 and 45. We notice that the former is closely parallel to the Old Babylonian AO 8862 No. 1 (translated above, section II), the difference between the two amounting to a permutation of addition and subtraction. The reference to the "number of sides" shows that the text is mixed up with one of the problems dealing with a rectangle and its four sides (Nos. 43 and 46), a corruption which is also clear from the ensuing numerical calculations (which is the reason why I have omitted the end of the problem). But already the beginning of the procedure shows that a shift of variable is intended, analogous to that of the Old Babylonian problem and reducing the problem to that of $L{\cdot}w=60$, $L-w=4$ ($L=l+2$). A similar reduction is performed in No. 45, where the whole procedure stands uncorrupted. It turns out to be precisely that of the Old Babylonian texts, using semi-sum and semi-difference.

This is a common feature of the first part of the *Liber Mensurationum*. In contrast, the Seleucid standard method makes use of full sum and difference (see above, section III). This supports the impression coming from the rhetorical structure of the problems (and that given by "the four sides") that the first half of the *Liber Mensurationum* is mainly affiliated directly to the Old Babylonian tradition, bypassing the Seleucid mathematicians, both regarding rhetorical and pedagogical build-up and as far as mathematical contents and method is concerned.

The surprising use of the term quadratus suggests that the translation is very conscientious and literal.[19] It should therefore be meaningful to submit Gherardo's text to precise terminological analysis in order to see to which extent the old Babylonian conceptual distinctions are still conserved.

It turns out that the disctinction between the "multiplicative" operations "raising", "making span" and "times" have been lost over the centuries. Even in the case of additions a certain loosening of the strict language is visible (the square which is "added" (Latin *addare*) to the number in No. 3 is "joined" (*adjungare*) in most others). Still, there are a number of preferred modes of expression which correspond well with Old Babylonian ways. *Adjungare* for *waṣābum*, "append", is one of them, *aggregare* for *kamārum*, "accumulate", is another (even though this Latin word is also used for other, non-additive processes). "Add ... over" corresponds precisely to *eli ... watārum*, an expression rendered freely as "exceed ... by" in section II. In several places one finds *ponare*, "put down", where Old Babylonian texts would have a *šakānum* with precisely the same meaning. When a geometric interpretation of a procedure calls for a concrete repetition (e.g., in case of the two rectangular surfaces in Figure 6) the term *duplare* (translating apparently *ḍa'ufa*) occurs, while Old Babylonian texts would have *eṣēpum* (so in Nos. 47 and 48). In No. 57 it is even told that in order to find the result of a quadruplation (*raba'a?*) you have to multiply by 4; certain Old Babylonian texts contain similar double constructions (in AO 8862 No. 1 a case of "making span" followed by "times" was found). Evidently, quadruplation must be understood as something different from arithmetical multiplication — and in the problem in question only the obvious possibility of concrete geometric repetition appears to be at hand.

[19] In the Boethian tradition as well as all the 12th-century Arabo-latin translations of the *Elements*, *quadratus* is invariably defined as an equilateral and right-angled quadrangle; see, Folkerts 1970 : 116, for the Boethian tradition; and van Ryzin 1960 : 81 (Adelard I), 148 (Adelard II), 199 (Adelard III), 274 ("Hermann"), 327 (Gherardo). The closest approach to the *quadratum altera parte longius* as designation of the rectangle is the Boethian *[quadrilaterus] altera parte longius* (Folkerts 1970: 116), while Adelard I has *quadratum longum* (van Ryzin 1960 : 81). It can hardly be doubted that Gherardo when translating Abû Bakr has tried to represent his Arabic text as faithfully as possible at the conditions of normal vocabulary and normal Arabo-Latin correspondences.

The statistical but not always absolute dominance of certain terms in certain connections suggests that some variant of the old naive-geometric procedures was still in use, but that it was described verbally in a language the terminological structure of which was not (or was no longer) fully adapted to its concrete procedures. In fact it is also evident, in Arabic as in other languages, that terms which originally designated concrete operations have gradually developed into technical terms for abstract arithmetical operations.

Some of Abû Bakr's problems have no counterpart in published Old Babylonian texts but have so in the Seleucid tablet BM 34568 (notably No. 47, mentioned in connection with Figure 6). But the terminology used in even these problems carries precisely those features which were just described, and it is quite far from the complete arithmetization of the Seleucid tablet. So, Old Babylonian or not, these problems too appear to have developed inside the mainstream of the tradition leading from Old Babylonia to Abû Bakr, most likely before the Seleucid branch split off; they have in all probability *not* been borrowed from the outside in the way a few problems of Greek inspiration seem to have been taken over.

As explained above, the treatise shares the "See!" with many Indian texts. At the same time it is obvious that both problems and procedures differ from the sophisticated Indian syncopated algebra. Since the word recurs so frequently in the first part of the treatise but not in the "Alexandrian" second part it is implausible that the usage can be a borrowing from India. Instead, it must belong with the mainstream development.[20] As it is strictly absent from the Old

[20] The presence of the term in India can then be interpreted either as the result of an isolated borrowing of a usage or as an indication that the development of Indian algebra was in its beginnings (from which it was to differentiate itself very creatively) influenced by the Babylonian tradition. A rule like this from Brahmagupta's *Kuṭṭaka* (quoted from Colebrooke 1817 : 347) for the solution of an equation $\alpha x^2 + \beta x = \gamma$ (presented as an alternative to the first rule which refers to the developed schemata) could indeed look like a borrowing from Old Babylonia, arithmeticized by the interaction with prevalent arithmetic ways of thought but still recognizable :

> To the absolute number multiplied by the [coefficient of the] square, add the square of half the [coefficient of the] unknown, the square root of the sum, less half the [coefficient of the] unknown, being divided by the coefficient of the square, is the unknown.

This is precisely the standard method of the Old Babylonian mathematicians for the solution of such equations, and it is better suited for geometric treatment than the current method of Medieval algebrists (reduction to $x^2 + (\beta/\alpha)\, x = \gamma/\alpha$ — see my (1985 : 14f).

Babylonian texts we can probably assume it to represent a change in the mainstream tradition taking place after Old Babylonian times.[21, 22]

All in all we may conclude that the first half of the *Liber Mensurationum* represents a tradition which goes back to Old Babylonian mathematics; which carries on the main features of the "rhetorical" structure of the Old Babylonian texts; and which was still making use of methods cognate to the naive geometry of the Babylonians when the Arabic original was formulated (but probably no longer when Gherardo made his translation). At the same time it presents us with an alternative, different, non-geometric tradition, identical in name and in contents with Al-Khwârizmîan *al-jabr*.

[21] Possibly in connection with the introduction of new material supports for drawings. In Old Babylonian teaching drawings may have been made in the sand of the school courtyard or on a dust-board (see my 1985 : 29); they are not on the clay tablets, which anyhow are not suited for stepwise alterations of figures, but whose texts can instead be read as *constructive prescriptions*. If later developments of the tradition were transferred to papyrus or some similar material, and if drawings were then aligned with the texts, less constructive formulations of the texts as well as a "see!" or "here is the figure" referring the reader to the drawing would be no wonder.

[22] Another puzzling connection between Abû Bakr's treatise and an old mathematical tradition is suggested by Nos. 33-34, asking for the sides of a rectangle of area 48 where furthermore $l/w = 1\ 1/3$ or $w/l = 3/4$, respectively (and by Nos 62-63, which raise the analogous problems for a rhomb). Apart from the value of the area, No. 34 coincides completely with problem No. 6 of Papyrus Moscow (see Struve 1930 : 125), and several other problems of the papyrus are related (Nos 7 and 17, *ibid.* pp. 128f and 133f). Moreover, the procedure is fundamentally the same in the two texts. On the other hand, the procedure in question, if not the simple problem itself, is also familiar from Old Babylonian texts, where it serves the solution of non-normalized *mixed* second degree-equations. Furthermore, the PM-problems themselves might be Babylonian borrowings — in contrast to the normal procedure of Egyptian mathematics, PM 6 and 17 perform their divisions the Babylonian way, through a multiplication by the reciprocal. (But independent invention is quite possible, the procedure consists in the intuitively simple comparison of the rectangle with an adequate square).

At closer inspection, Abû Bakr's problems 33-34 turn out to contain several formulations of Old Babylonian stamp. I would therefore confidently consider the Egyptian trace a red herring.

V. Augmentation and Diminution

Once the *Liber mensurationum* is known, it becomes obvious that Abraham bar Ḥiyya's (Savasorda's) *Collection on Mensuration and Partition* (*Ḥibbur ha-mᵉšihah wᵉha-tišboret*, in *Latin Liber Embadorum*, see the edition in Curtze 1902) is indebted to the same tradition for the part dealing with squares and rectangles (as both works depend on the Alexandrian tradition for other parts). Since Abraham uses the same procedures as Abû Bakr and demonstrates their correctness in a geometric explanation followed by words like "and this is the figure" and a drawing, his treatise gives us some support for the above interpretation of the word *intellige*. But Abraham draws directly on the *Elements* for his proofs instead of using naive manipulation of areas (the contents of II. 5 is quoted as trivial knowlegde in Curtze 1902:40[7] ff, that of II. 6 on p. 36[10]ff, II, and that of II. 7 on p. 42[18]ff). Evidence from his hand can therefore only claim a hypothetical bearing on questions concerned with early Islamic, sub-scientific mathematical traditions.

The same can be said on Leonardo Fibonacci's *Practica Geometriae*, which contains many of the same problems in the section on squares and rectangles (Boncompagni 1862:56-77). Leonardo goes one step farther than Abraham in his syncretism, mixing up the old problems both with Euclidean principles and with the vocabulary of *al-jabr*.

The most important fact about these two run-away descendants of the tradition is that they appear to be both mutually independent and independent of Abû Bakr. If so, the *Liber Mensurationum* must be regarded as a representative of a wide-spread tradition in his times, not as a last survivor from a dying environment (cf., also on Abû Kâmil in the following section).

Possibly the *Liber Mensurationum* contains a hint where to look for cognate works. In fact, I may be in error above when claiming that the basic method of the treatise is, in contrast with the "method according to *al-jabr*", unnamed. In No 9 it is said in the end of the basic procedure that this is "according to *augmentum et diminucionem*". Possibly, these words refer to the double root of the problems just solved, $s=2\pm\sqrt{2^2-3}$ (obtained as in the Old Babylonian AO 8862 No. 1, cf., section II). But the phrase is followed immediately by the sentence "Its method according to *al-jabr* is, however, that...", suggesting a contrast between *al-jabr* and *augmentum et diminutio:* if no such contrast is intended, the "however" (*vero*) is clumsy style of a sort not found elsewhere in the text. It therefore seems a reasonable guess (but no more) that the basic naive-geometric method carried the name "augmentation and diminution".

This would be no bad name. It is akin to "cut-and-paste" (or rather "paste-and-cut"), a pet-name which I have often used to characterize the Old Babylonian transformation of areas. But what would then be the corresponding Arabic name?

Evidently, Abû Bakr's expression is not to be confounded with that of the title *Liber Augmenti et Diminutionis,* a work explaining calculation with double false position (Libri 1838:I, 304-371). But since Woepcke's old conjecture (1863:514) concerning identity of this with the missing *Kitâb fi'l-jam' wa'l-tafrîq* has been convincingly rejected by Ruska (1917:15 f), it could perhaps be assumed that precisely this recurrent title could cover works in the Abû-Bakr-tradition. After all, the original meaning of *jama'a* appears to be concrete accretion, aggregation and completion (~~— Hebrew ʾm', "festigen", "[Kind] großziehen", "[Haus] restaurieren"??~~), while that of *faraqa* is "to sunder" (cf., Akkadian *parāqum*, "abtrennen", and Hebrew prq, "ablösen", "wegnehmen").

(nonsense, the Hebrew word is ʾmṣ)

Sezgin lists several treatises dealing with *al-jam' wa'l-tafrîq.*[23] Unhappily, all the works in question are known only from Al-Nadîm's *Fihrist*; all that can be seen is therefore that the subject was cultivated by persons also interested in *al-jabr,* Hindu reckoning or "calculation" (*ḥisâb*), and that it disappeared as a subject for independent treatises in the early 4th/10th century.

The argument which Ruska used to reject Woepcke's identification of the subject can be used against his own identification with "Hindu reckoning"; in fact, Abû Ḥanifa is told by Al-Nadîm to have written separate treatises on the two subjects.

[23] See, GAS V, 227^{1f} (Aḥmad ibn Muḥammad al-Ḥâsib, "the calculator"); 243^{10} (misplaced in Flügel's *Fihrist* edition under Sind ibn 'Alî, belongs probably with Al-Khwârizmî, cf. note 3); 263^{1} (Abû Ḥanîfa al-Dînawarî); 281_{6} (Abû Kâmil); 301^{12} (a commentary on Al-Khwârizmî's treatise written by Al-Ṣaidanânî); and $301_{11,9}$ (an independent treatise and a commentary on another treatise by Sinân ibn al-Fatḥ).

The archaic terminology of the *Liber Mensurationum* would place it in a a rather early epoch,— 3d/9th century at the latest, I suppose, well before Abû Kâmil, Al-Ṣaidanânî and Sinân ibn al-Fatḥ. So, the *Fihrist* can at least be claimed not to contradict the hypothetical identification of Abû Bakr's method with *al-jam' wa'l-tafrîq*; but its support for the hypothesis is at best vague and uncommitting. Confirmation or rejection must await stray finds in texts or libraries.[24]

VI. Other Witnesses: Thâbit and Abû Kâmil

After this walk on thin or non-existent ice we shall return to firmer ground, first to Thâbit's treatise "on the rectification of the cases of *al-jabr*" (*fî taṣḥîḥ masâ'il al-jabr*; in Luckey 1941).

The "cases of *al-jabr*" are treated through its three "elements" (*uṣûl*), coincident with Al-Khwârizmi's 4th, 5th, and 6th case but numbered from 1 to 3. The geometric proofs are also performed in (real or feigned) ignorance of Al-Khwârizmî's justifications. Further, the subject is labelled as stated, *not* as *al-jabr wa'l-muqâbala.* Finally, the subject is apparently not that of a book but one belonging with a group of practitioners, the "*al-jabr*-people" (*ahl al-jabr*) or "followers of *al-jabr*" (*aṣḥâb-al-jabr*). If we think of the short span of time which separates Al-Khwârizmî and Thâbit (leaving no time for such a community to develop from scratch nor, *a fortiori,* to repress the memory of its founding father) it is clear that the company of *al-jabr* must be a group which was *not inspired by Al-Khwârizmî*; instead it *supplied him with inspiration.*

A further look at the text makes it clear that *al-jabr* as known to Thâbit is strictly identical with the discipline known to Abû Bakr under the same name. Hence, the Al-Khwârizmîan numbering of the fundamental cases in the *Liber Mensurationum* cannot be taken as evidence that Abû Bakr is really inspired by Al-Khwârizmî.

In Abû Kâmil's *Algebra,* the idea of a special group of *al-jabr*-people seems to have disappeared. Instead, the subject *is* now understood as the discipline of Al-Khwârizmî's *Kitâb fî al-jabr wa'l-muqâbala* (see the text in Levey 1966:28 f, including notes 1-2). There are, however, passages where a plurality of distinct traditions are spoken

[24] In his contribution to the seminar, Ahmad Selim Saidan suggested the alternative hypothesis that *al-jam' wa'l-tafrîq* designates advanced arithmetic based on finger-reckoning.

of, namely problems No 7 and 8 (Levey's counting). In No 7 (Levey 1966:92-95), the number 10 is to be divided into two parts of which one is taken as the *thing* and the other as 10 minus the *thing*; this is well-known both from Al-Khwârizmî and from Abû Bakr's *al-jabr*-methods. *Alternatively*, the semi-difference between the two numbers is taken as the thing, and this way is referred to the "possessors of number" (bʿly h-mspr in the Hebrew text). This alternative looks as an *al-jabr*-interpretation (because of the "thing" and the corresponding arithmetico-rhetorical unfolding of the argument) of the steps of the Old Babylonian method.[25] It would then be reasonable to take it as Abû Kâmil's interpretation of Abû Bakr's basic method in his own conceptual framework.

In No. 8, which also divides the number 10 into two parts (Levey 1966:94-103), it is the *al-jabr*-method (one number taken as the "thing") which is ascribed to a particular group, "those who pursue calculation" (ynhgw h-ḥšbnys). The closeness of Hebrew ḥšb and Arabic *ḥisâb* makes it fairly sure that Abû Kâmil spoke of people engaged in *ḥisâb*, practical commercial arithmetic, accounting, etc. Astronomers or other scientific practitioners can hardly be meant.

These two references to groups of traditional sub-scientific mathematical practitioners are the only ones contained in Abû Kâmil's work, although he can be seen to draw on the methods of such environments in other places without indicating his source (see, Anbouba 1978:75, 82f). The subject is referred to Al-Khwârizmî, and it is given the full name of his presentation of the subject, *al-jabr wa'l-muqâbala*. At the same time the meaning of the term is widened, from the *al-jabr* of the *Liber Mensurationum* to that of *algebra* in our sense. When Abû Kâmil was writing (early 4th/10th century?) the separate sub-scientific traditions were, at least when seen from Abû Kâmil's perspective, in the end of a process of absorption and integration with *mathematics* understood as a unified field ranging from high-level science to low-level but still reasoned and correct applications.[26] Even when considered as algebrists the mathematical

[25] It is also Diophantos' method. But since this author was not known to Abû Kâmil we should not expect him to have a whole host of followers in Abû Kâmil's environment.

[26] This general unification of Islamic mathematics and its cultural background is the main subject of my (1984). In reality the process was well under way but not nearly completed in the 4th/10th century.

practitioners of Islâm were becoming a "people of the Book", — and so, witnesses later than Abû Kâmil cannot be expected to have had access any longer to a situation similar to that encountered by Al-Khwârizmî and Ibn Turk who *Wrote the Book*.[27]

VII. Al-Khwârizmî and Ibn Turk

Let us therefore return to these founding fathers, — first to Al-Khwârizmî, whose ample treatise offers more opportunity for analysis than the short fragment surviving from Ibn Turk.

[27] If we are willing to transgress the borders of *al-jabr*, some interesting information can be gained from Abû'l-Wafâ's *Book on What is Necessary from Geometric Construction for the Artisan* (*Kitâb fî mâ yaḥtaj al-ṣâniʿ min al-aʿ mâl al-handasiyya*), written after A.D. 990 according to its dedication. In chapter 10, proposition 13, the author tells that he has taken part in certain discussions between "artisans" (*ṣunnaʿ*) and "geometers", apparently regarded as more or less coherent professions. Confronted with the problem of adding three geometric squares (the sum also being a square), the artisans proposed a number of solutions, "to some of which were given proofs", — proofs which turn out to be of cut-and-paste-character. The geometers too had provided a solution (in Greek style), but that was not acceptable to the artisans, who claimed a concrete rearrangement of parts into which the original squares could be cut. So, Abû'l-Wafâ' confirms directly several of the indirect inferences from the *al-jabr*-texts : The environment of practitioners carried on its own mathematical tradition; this tradition was, at least in part, supported by geometric proofs; but the style and the basis both of its proofs and of a number of procedures were explicitly different from those of Greek geometry, and related to the ones described in the Old Babylonian texts. (See, pp. 115ff in S.A. Krasnova's translation of the work, in A.T. Grigor' jan — A.P. Juskevič (eds), *Fiziko-matematičeskie nauki v stranax Vostoka* I (IV), 42-140 (Moscow 1966), translating fols 53ff of the Istanbul manuscript (Aya Sofya, 2753). Cf., also pp. 348ff in Woepcke, "Analyse et extrait d'un recueil de constructions géométriques par Aboûl Wafâ", *Journal Asiatique*, 5e série *5* (1855), 218-256, 309-359. quoting and paraphrasing the Persian Paris manuscript (BN, pers. anc., 169).

It can be observed that many of Abû'l-Wafâ's problems begin with the Old Babyloian "If he says" (while the eclectic character of the work is revealed by subsequent use of the Greek-styled "we" in the prescriptions). It is also to be noticed that the prescriptions end with an invariable "This is the figure".

In 1969 Kubesov and Rosenfeld pointed out (*Archives Internationales d'Histoire des Sciences 22*, 50) that large parts of Abû'l-Wafâ's text are taken over directly from Al-Fârâbî's *Book on Spiritual Ingenuities and Natural Mysteries about Subtleties of Geometrical Figures.* It would certainly be interesting to make a close investigation of the procedures and formulations contained in this work, which was finished already in A.D. 933.

Al-Khwârizmî's starting point is *al-jabr*, *not* the basic method of the *Liber Mensurationum*. This is clear already from his use of the "cases", from his use of the terms *mâl* ("wealth") and *jadr* ("root"), and from the subsequent arithmetico-rhetoric organization of the argument around the *shay* ("thing"). The Greek-tainted naive-geometric justifications are, already from their own formulation and appearance, grafted upon the main line of the book (and now when the existence of a naive-geometric tradition is certified we may assume with fair certainty that they were taken over from there). The secondary character of the geometric justifications is still more clear when the addition of

(100+wealth+20 roots) and (50+10 roots+2 wealths)

is discussed (Rosen 1831:33f). Here the author confesses that he has "contrived to construct a figure also for this case, but it was not sufficiently clear", while the "elucidation by words is very easy" and given rhetorically.

In the fragment of Ibn Turk's treatise the same basic orientation of thought in agreement with the *al-jabr*-pattern is also visible. Here too we have the standard cases, and here too they are defined in terms of *mâl* and *jadr*, not through the "area" and "side" which are the fundaments of the ensuing geometric justifications.

In Ibn Turk we find, however, a more outspoken parallel similarity with the naive-geometric tradition as reflected in the *Liber Mensurationum* than in the case of Al-Khwârizmî. A square is indeed not simply a *murabba'* to Ibn Turk but an "equilateral and equiangular *murabba'*".)The same usage is found only occasionally in Al-Khwârizmî, who in most places writes simply *murabba'* (see, Sayılı 1962:84). [28]

[28] This observation influences the question of priority and dependence. When Ibn Turk is so much closer than Al-Khwârizmî to the original use of a central term in the naive-geometric tradition, he can hardly have taken over his ideas from Al-Khwârizmî. Since the existence of two living traditions makes independent combination possible we cannot, on the other hand, conclude from here that Al-Khwârizmî copied Ibn Turk. Nor can we be sure that his writings are later. Most likely, the value of *murabba'* was changing first in the circle of court mathematicians around Al-Ma'mûn, a place where the Greek influence was probably stronger than elsewhere, and the very environment in which Al-Khwârizmî wrote his book. After all, the best literal translation of Greek τετράγωνον, "square", is nothing but *murabba'*.

Another similarity with the *Liber Mensurationum* is more equally shared between the two. Both authors end their geometric explanations by a "This is the figure" (Al-Khwârizmî) or "And this is the shape of the Figure" (Ibn Turk), — precisely as it was also found in Abraham bar Ḥiyya's *Collection*.

So, we are led to the conclusion that both authors supplemented their treatise on the methods of the "*al-jabr*-people" with material borrowed from another sub-scientific tradition. They did so, however, from a conception of mathematics foreign to both sub-scientific traditions (as far as it can be judged from the indirect evidence at hand), namely from the idea that mathematics should be supplied with proofs. [29] This, and not only the use of letters to identify geometric entities and the way to explain the construction of a geometric figure, was in the scientific mathematical tradition initiated by Greeks. The fundamental feat of the two authors was to bring the two levels of mathematical activity together for mutual fructification.

Appendix I: Al-Jabr

Since the Middle Ages much ink has been used in discussions of the meaning of the terms *al-jabr* and *al-muqâbala*. The reinterpretation of Babylonian mathematics and the recognition of the *Liber Mensurationum* as a source for early Islamic mathematics raises the question anew and supplies us with new evidence for the origins of the terms, and hence maybe with information concerning the history of the art (but of course not with evidence for the interpretation of the terms in mature Islamic mathematics, [30] in agreement with the definition of etymology as the "lore of no longer valid meanings").

In this connection, the Adelard-I-term for a rectangle (see, above, note 19) may be of interest. It seems to bear witness of a time when the change of value was not completed (but the definition of *quadratus* suggests that it was on its way). Now, Adelard I appears to have been made on the basis of a version of Al-Ḥajjâj's second translation, performed at al-Ma'mûn's court (see, Busard 1983: 18f, and Murdoch 1971: 445).

[29] It is precisely the lack of *explicit and autonomous* interest in proof (as distinct from practical and only implicit *understanding*) which makes me speak of *sub*-scientific traditions.

[30] For this question, I shall only refer to Saliba's discussion (1972) of the confusing use and the contradictory definitions of the terms in a variety of Islamic authors.

A few years before the discovery of Babylonian second-degree algebra Gandz suggested (1926) that muqâbala should be seen as a secondary term, repeating in Arabic an Aramaic descendant of the Akkadian term *gabrum*, "opponent", "equivalent" etc. In Gandz's opinion, this term could have been used to designate equality, and hence "equation", in an Assyrian ancestor of algebra.

Now, *gabrûm* is originally a Sumerian loan-word (from gabari); the sumerogram can be used ideographically for the verb *maḫârum* and its derivations. The latter word is in fact important in Old Babylonian second-degree algebra — cf., above, section II. Its function is not, however, that supposed by Gandz; instead it has to do with the formation of a square from its equal sides. Since early Islamic *al-jabr* is indeed second-degree arithmetico-rhetoric algebra, it is not implausible that the term can have followed the art as it rambled down the ages. At a time when the conceptualization was arithmeticized practitioners of the field may well have re-interpreted the term[31] — this could easily happen since the loan-word spread to other Semitic languages, including Aramaic and Hebrew (where gbr possesses a double meaning analogous to the English *peer*). *Muqâbala* can then have been appended to the name as an explanation (which need not have happened in the Arabic phase; the same root in closely related meanings is found in other Semitic languages, from Akkadian and Aramaic to Ethiopian).

At this point, the *Liber Mensurationum* comes in. In fact, several of its expositions "according to *al-jabr*" contain implicit definitions of the two operations *al-jabr* and *al-muqâbala*, translated as *restauratio* and *oppositio*. Let us, e.g., look at No. 5, where "wealth minus thing" (*census excepta re*) equals 90. Then "restore and oppose, that is that you restore the wealth by the subtracted thing and add it to the 90, and you will have a wealth which equals a thing and 90 drachmas." In No. 7 it is asked how to "restore 2/5 of 1 so that you get 1", and the answer is that you multiply by 2 1/2. So *al-jabr* covers restoration both by addition and by multiplication, in agreement with the meanings testified in other texts. *Al-muqâbala* is, on the other hand, different from the normal meaning of most later texts (where it implies the dropping of similar terms on both sides of the equation); it designates the opposition of two different quantities of equal magnitude, i.e., the formation of a ("rhetorical") equation. (This is perhaps even more evident in a passage of No. 7 where only an "opposition" and no "restoration" is performed). In this respect, Gandz's has therefore proved fully correct[32] (irrespective of the correctness of the supposition of an translation of *gabrûm*). On the other hand, "restoration" and "opposition" are obviously names for *different* operations even in the *Liber Mensurationum* (as in Al-Khwârizmî and others). If *qabila* has been brought in at some moment as an explanation of a descendant of *gabrûm* this must have happened long before the times of Abû Bakr — and in consideration of his archaic terminology it must in all probability have happened before the Arabic became the carrying language of the art.[33]

[31] "Restoration" (cf. below) will then have been a second reinterpretation.

Appendix II: Successive Doublings

Many interesting conclusions follow from the discovery of a direct tradition leading from the Old Babylonian scribal school to the *Liber Mensurationum*. But cautious doubts may remain: Is the existence of such a silent tradition over several millenia not still more improbable than the random repetition of phrases and rhetorical structures?

The existence of another continuous sub-scientific mathematical tradition over the same span of time may put the doubt to rest.

In the last chapter of the last book of his large explanation of Hindu reckoning, Al-Uqlîdisî states that this is a question many people

[32] Incidentally, the term is *used* in the same way by another Abû Bakr, *viz.* by Abû Bakr Muḥammad ibn al-Ḥasan al-Karajî, in his *Fakhrî* (Woepcke 1853:64) as well as *Badi'* and *Kâfî* (according to Saliba 1972: 199f; the *definition* in the *Kâfî* appears to be Al-Khwârizmîan, see, Hochheim 1878: III, 10).

[33] Further discussions of the problem should take into consideration the origins of the terms *mâl* (which, when used as the unknown quantity of a first-degree problem corresponds to the θησαυρός of the Greco-Egyptian Papyrus Akhmîm — see, Baillet 1892: 70, 72) and *jadr*. The latter metaphor (which considers x the "cause" of x^2, or rather $\sqrt{x}$ the "cause" of x) is already testified in India in the first century B.C. (see, Datta and Singh 1962: 169f). A diffusion of the idea via Iran is plausible — and in that connection it may become interesting that both Ibn Turk and Al-Khwârizmî are of Turkestanian descent (as already pointed out in Sayılı 1962:87f).

The mixed evidence suggests a number of cross-fertilizations rather than unilinear descent and one-way diffusion.

ask. Some ask about doubling one 30 times, and others ask about doubling it 64 times" (Saidan 1978:337). A little later in this text from the year 341/952, 10 successive doublings are discussed in a way reminding very much both of a Seleucid text and of later Islamic arithmetical textbooks.

Evidently, the 64 doublings are identical with the classical Indian chess-board problem. The 30 doublings are found as No 13 in a late 8th or early 9th century (A. D.) Latin problem collection ascribed to Alcuin (in Folkerts 1978; No. 13 is pp. 51f), formulated thus:

> A certain king ordered his servant to collet an army from 30 cities by taking from each city as many men as he brought to it. But he came alone to the first city, and brought another with him to the second; now, three came [with him] to the third. Let him who can say how many men were collected from these 30 cities.

I have discussed the widespread occurences of these doublings in my (1984:10) as one illustration among others of the sub-scientific commercial and recreational mathematical tradition shared in late Antiquity and in the early Middle Ages along the Silk Route, from China to Western Europe. Quite recently, however, an Old Babylonian text from Mari was published (in Soubeyran 1984: 30) which sheds some astonishing new light into the matter. It runs as follows:

1 grain has appended 1 grain:
2 grains the first day.
4 grains the 2nd day.
8 grains the 3d day.
16 grains the 4th day
. . . .
and so on until 30 days.

Firstly, the number of doublings is one of those asked for "by many people" in 4th/10th century Damascus, and the one asked for in the Carolingian problem collection. Secondly, it describes the doublings in the same additive manner as the latter text. Thirdly, it deals with grains of wheat or barley. What might look before as two different but analogous recreational problems meeting in Damascus seems now to be members of the same old family. In any case, the relationship between the Mari problem and the Carolingian problem (separated by 2500 years) seems established.

The consanguinity of the Old Babylonian and the Indian problem is supported by another observation. Game boards and calendaric boards with 3·10 fields corresponding to the 30 days of the month have been excavated in several Ancient Middle Eastern sites. One, from Habuba Kabira (quite close to Mari) and dating from the late 4th millenium B.C., was shown at an exhibition in the FRG in 1980 and 1981. Others, from 2nd millenium Susa and Palestine, were published by de Kainlis (1942:27f, discussion pp. 33f).[34] They establish a plausible connection between the 30 days and the game-board, and thus another link between Mari and the chessboard.

Appendix III: Al-Khwârizmî's Geometric Justification of the Case "Roots and Squares are Equal to Numbers", Reprinted from Rosen 1831: 13-16.

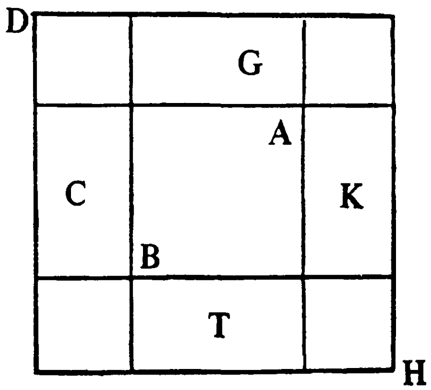
DEMONSTRATION OF THE CASE: "A SQUARE AND TEN ROOTS ARE EQUAL TO THIRTY — NINE DIRHEMS"*

The figure to explain this a quadrate, the sides of which are unknown. It represents the square, the which, or the root of which, you wish to know. This is the figure AB, each side of which may be considered as one of its roots; and if you multiply one of these sides by any number, then the amount of that number may be looked upon as the number of the roots which are added to the square. Each side of the quadrate represents the root of the square; and, as in the instance, the roots were connected with the square, we may take one-fourth of ten, that is to say, two and a half, and combine it with each of the four sides of the figure. Thus with the original quadrate AB, four new parallelograms are combined, each having a side of the quadrate as its length, and the number of two and a half as its breadth; they are the parallelograms C, G, T, and K. We have now a quad-

[34] Dr. Peter Damerow told me about the Habuba Kabira board; Professor Wolfram von Soden referred me to de Kainlis. I am grateful to both for their assistance.

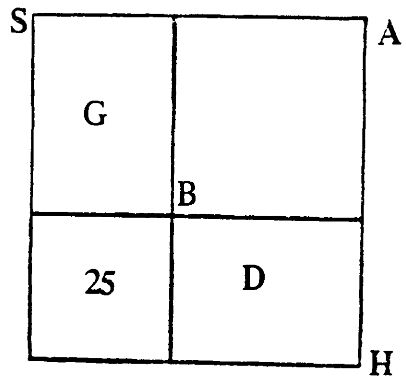
* Geometrical illustration of the case, $x^2 + 10x = 39$

rate of equal, though unknown sides; but in each of the four corners of which a square piece of two and a half multiplied by two and a half is wanting. In order to compensate for this want and to complete the quadrate, we must add (to that which we have already) four times the square of two and a half, that is, twenty-five. We know (by the statement) that the first figure, namely, the quadrate representing the square, together with the four parallelograms around it, which represent the ten roots, is equal to thirty-nine of numbers. If to this we add twenty-five, which is the equivalent of the four quadrates at the corners of the figure AB, by which the great figure DH is completed, then we know that this together makes sixty-four. One side of this great quadrate is its root, that is, eight. If we subtract twice a fourth of ten, that is five, from eight, as from the two extremities of the side of the great quadrate DH, then the remainder of such a side will be three, and that is the root of the square, or the side of the original figure AB. It must be observed, that we have halved the number of the roots, and added the product of the moiety multiplied by itself to the number thirty-nine, in order to complete the great figure in its four corners; because the fourth of any number multiplied by itself, and then by four, is equal to the product of the moiety of that number multiplied by itself.* Accordingly, we multiplied only the moiety of the roots by itself, instead of multiplying its fourth by itself, and then by four. This is the figure:



* $4 \times (b/4)^2 = (b/2)^2$

The same may also be explained by another figure. We proceed from the quadrate AB, which represents the square. It is our next business to add to it the ten roots of the same. We halve for this purpose the ten, so that it becomes five, and construct two quadrangles on two sides of the quadrate AB, namely, G and D, the length of each of them being five, as the moiety of the ten roots, whilst the breadth of each is equal to a side of the quadrate AB. Then a quadrate remains opposite the corner of the quadrate AB. This is equal to five multiplied by five: this five being half of the number of the roots which we have added to each of the two sides of the first quadrate. Thus we know that the first quadrate, which is the square, and the two quadrangles on its sides, which are the ten roots, make together thirty-nine. In order to complete the great quadrate, there wants only a square of five multiplied by five, or twenty-five. This we add to thirty-nine, in order to complete the great square SH. The sum is sixty-four. We extract its root, eight, which is one of the sides of the great quadrangle. By subtracting from this the same quantity which we have before added, namely five, we obtain three as the remainder. This is the side of the quadrangle AB, which represents the square; it is the root of this square, and the square itself is nine. This is the figure:—



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